(*Paper No. 2110.*)

"The Design and Stability of Masonry Dams."

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This Paper is founded on a collection of notes made by the Author whilst recently engaged in designing some masonry dams for the Rio Tinto Mining Company in Spain.

The subject has been divided into two parts. (1) The Design, and (2) the Calculations of Stability. This latter part of the subject is a comparatively simple matter, and by means of the equilibrium-polygon of graphic statics the conditions of stability of any given profile are easily ascertained.

Owing to the indeterminate nature of the problem, it seems impossible to construct a general formula for calculating the dimensions of a dam, and the method usually followed consists in assuming an approximate profile, and then testing its stability by a graphic resolution of forces. If found defective the profile is altered, and the graphic process repeated until a sufficiently exact result is obtained. In large dams a continued repetition of trial alterations becomes very troublesome, and in order to avoid this difficulty the Author devised the method given in Part I of this Paper, by means of which a profile can be calculated with considerable accuracy. The profile thus obtained may then be tested by the graphic process given in Part II, when it will be found that if any alteration is required it will be only in the lower portion of very high dams.

In what follows, the two faces of the dam will be called respectively the "inner face" and the "outer face"—the "inner face" being that against which the water rests. The inner face will for convenience sometimes be called the "back" of the dam.

I.

There are two conditions of stability upon which the design of a dam must be based.

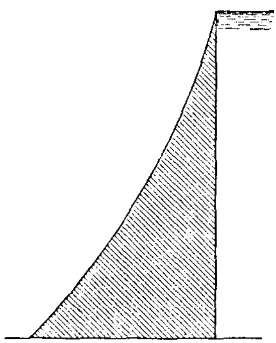
(1) The resultant of all the forces acting at any horizontal section must fall within the middle third of the thickness of the dam.

(2) The stresses in the faces of the dam must not exceed the safe limit.

These two conditions must be satisfied for the extreme cases of reservoir full, and reservoir empty.

Condition (1) implies that there should be no tensile-stress in the masonry, and this is especially necessary for the full reservoir. Another condition might be added, namely, that at any horizontal section the dam must be safe against sliding, but practically this is always the case when conditions (1) and (2) are satisfied.

FIG. 1.



One of the latest works on the theory of dams is a Paper by Mr. Pelletreau, published in the Annales des Ponts et Chaussées for 1876–77[1], in which a formula is given for the curve of the outer face of the dam with a vertical back, and a thin crest at the water-level (Fig. 1). The profile given by this formula fails in condition (1), and the same objection applies to all similar calculations which are based on condition (2) alone, and in which it is sought to obtain a "profile of equal resistance"—that is, a profile with a constant stress in the faces. Mr. Pelletreau's formula also fails in condition (2), as it does not take into account the effect of the obliquity of the resultant, which has been shown by Mr. Bouvier[2] to be of considerable importance.

In commencing the design of a profile, it is necessary to determine beforehand the width of the top, and its height above water-level. Both these dimensions depend to some extent on local circumstances. As a general rule the Author considers the values given by the following empirical formulas to be sufficient (Fig. 2).

For English feet—

$$x_0 = 4\cdot 0 + 0\cdot 07\,\mathrm{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$y_0 = 1\cdot 8 + 0\cdot 05\,\mathrm{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where x_0 is the width of the top of the dam, y_0 the height above water-level, and H the total height of the dam (*i.e.* the greatest depth of water).

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[1] 5e série. Tome xii. p. 536; tome xiv. pp. 258, 480. [2] *Ibid.* Tome x. p. 173.

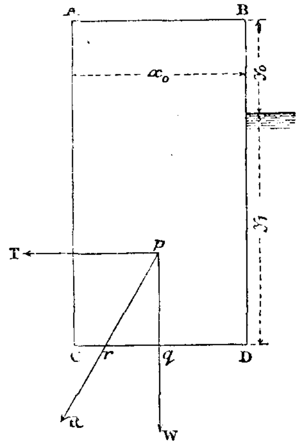
For metres these equations are:—

$$x_0 = 1 \cdot 22 + 0 \cdot 07\ \mathrm{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$y_0 = 0 \cdot 55 + 0 \cdot 05\ \mathrm{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

In the following calculations the dam will be considered to be a unit in length, measured in a direction perpendicular to the plane of the paper.

Fig. 2.



Let A B C D, Fig. 2, represent a cross-section of the top portion of a dam, x_0 and y_0 having been calculated by equations (1) and (2). It is evident that for a certain depth y_1 below the water, both faces may be vertical, and it would be easy to show that this depth must be determined by condition (1). That is, the resultant of all the forces acting at the section D C must fall within the middle third of D C.

These forces are—

W = the weight of the portion A B C D acting through its centre of gravity, and bisecting D C at q;

T = the horizontal thrust of the water acting at a distance $\frac{y_1}{3}$ above D C.

Calling ϕ the weight of a cubic unit of water, and ϕ' the weight of a cubic unit of masonry—

$$\mathrm{W} = \phi' x_0 (y_0 + y_1),$$

and

$$\mathrm{T} = \frac{\phi y_1^2}{2}.$$

Let R = the resultant of W and T, acting through their intersection p and cutting the base D C at r.

Then

$$\frac{q\,r}{p\,q} = \frac{\mathrm{T}}{\mathrm{W}} = \frac{\phi y_1^2}{2 \phi' x_0 (y_0 + y_1)}$$

$$q\,r = p\,q \cdot \frac{\phi y_1^2}{2 \phi' x_0 (y_0 + y_1)};$$

but
$$p\,q = \frac{y_1}{3}$$
therefore
$$q\,r = \frac{\phi}{\phi'} \cdot \frac{y_1^3}{6\,x_0\,(y_0 + y_1)}.$$

By condition (1) $q\,r$ must not exceed $\frac{x_0}{6}$; therefore, putting—

$$\frac{\phi}{\phi'} \cdot \frac{y_1^3}{6\,x_0\,(y_0 + y_1)} = \frac{x_0}{6};$$

and reducing—

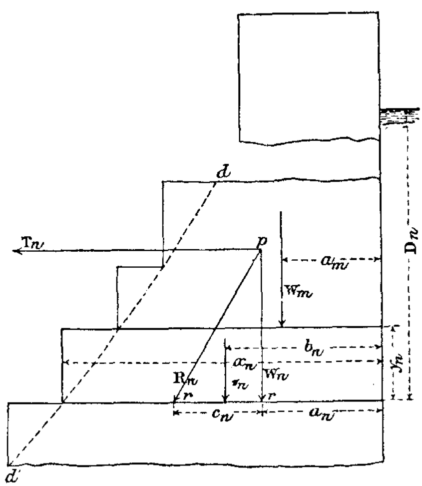
$$y_1^3 - y_1 \frac{\phi'}{\phi} x_0^2 - \frac{\phi'}{\phi} x_0^2\,y_0 = 0.$$

Denoting $\frac{\phi'}{\phi}$ by θ, where θ is the specific gravity of the masonry, the last equation becomes—

$$y_1^3 - y_1\,\theta\,x_0^2 - y_0\,\theta\,x_0^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the value of y_1 which satisfies this equation is the greatest admissible under condition (1).

Fig. 3.



In calculating y_1 it might be advisable to assume the water to be level with the top of the dam, so as to include the effect of a possible rise of water in a flood, or an increase of pressure from waves. In this case by making $y_0 = 0$ equation (3) becomes

$$y_1 = x_0\,\sqrt{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

Below y_1 the outer face must be curved for some distance down. The back will still be assumed to be vertical, and in order to simplify the calculations the dam will be considered to consist of a series of rectangular layers as shown in Fig. 3.

Let x_n (Fig. 3) be the width of any layer, n and y_n its height. Also let—

W_m = the weight of the dam above the layer n;
a_m = the distance of W_m from the back of the dam;
π_n = the weight of the layer n;
b_n = the distance of π_n from the back of the dam;
W_n = the resultant of W_m and π_n;
a_n = the distance of W_n from the back of the dam;
D_n = the depth from water-level to the base of the layer n;
T_n = the horizontal thrust of the water due to the depth D_n;
R_n = the resultant of W_n and T_n;
ϕ and ϕ' as before are the weights of a cubic unit of water and of masonry respectively.

Let q and r be the points where W_n and R_n cut the base of the layer n, and let the distance $q\,r$ equal c_n.

Taking moments about the back of the dam—

$$W_n\, a_n = W_m\, a_m + \pi_n\, b_n$$

$$a_n = \frac{W_m\, a_m + \pi_n\, b_n}{W_n};$$

but
$$\pi_n = \phi'\, x_n\, y_n$$

$$b_n = \frac{x_n}{2};$$

and
$$W_n = (W_m + \pi_n) = W_m + \phi'\, x_n\, y_n$$

therefore
$$a_n = \tfrac{1}{2}\left[\frac{2\, W_m\, a_m + \phi'\, x_n^2\, y_n}{W_m + \phi'\, x_n\, y_n}\right] \quad . \quad . \quad . \quad (a)$$

In the triangle $p\,q\,r$—

$$\frac{q\,r}{p\,q} = \frac{T_n}{W_n}$$

therefore
$$q\,r = p\,q \cdot \frac{T_n}{W_n} = c_n;$$

but
$$p\,q = \frac{D_n}{3},$$

and
$$T_n = \frac{\phi\, D_n^2}{2};$$

therefore
$$c_n = \frac{\phi\, D_n^3}{6\, W_n}$$

$$= \frac{\phi\, D_n^3}{6\,(W_m + \phi'\, x_n\, y_n)} \quad . \quad . \quad . \quad (b)$$

By condition (1) $(a_n + c_n)$ must not exceed $\frac{2}{3} x_n$; therefore, adding (a) and (b), and equating with $\frac{2}{3} x_n$—

$$\frac{3\,(2\,W_m\,a_m + \phi'\,x_n^2\,y_n) + \phi\,D_n^3}{6\,(W_m + \phi'\,x_n\,y_n} = \frac{2\,x_n}{3},$$

and reducing—

$$x_n^2 + \frac{4\,W_m}{\phi'\,y_n} \cdot x_n - \frac{6\,W_m\,a_m + \phi\,D_n^3}{\phi'\,y_n} = 0$$

$$x_n = \sqrt{4\left(\frac{W_m}{\phi'\,y_n}\right)^2 + \frac{6\,W_m\,a_m}{\phi'\,y_n} + \frac{\phi\,D_n^3}{\phi'\,y_n}} - \frac{2\,W_m}{\phi'\,y_n}.$$

Substituting in this equation $A_m\,\phi'$ for W_m, where A_m is the area of the dam above the layer n, and putting as before $\frac{\phi'}{\phi} = \theta$, it becomes—

$$x_n = \sqrt{4\left(\frac{A_m}{y_n}\right)^2 + \frac{6\,A_m\,a_m}{y_n} + \frac{D_n}{\theta\,y_n}} - \frac{2\,A_m}{y_n} \quad . \quad . \quad (4)$$

Making y_n = unity; equation (4) becomes—

$$x_n = \sqrt{4\,A_m^2 + 6\,A_m\,a_m + \frac{D_n^3}{\theta}} - 2\,A_m \quad . \quad . \quad . \quad (4a)$$

Commencing therefore with the top, the thickness of the dam at each successive layer can be calculated by equations (4) or (4 a). This gives a stepped form to the outer face, and by drawing a line as $d\,d'$ (Fig. 3), joining the inner angles of the steps, a practically correct curve is obtained.

There is a peculiarity about the outer face which greatly facilitates its design, namely, that towards the middle of the height its inclination becomes constant; so that when during the process of calculation this is found to occur, the use of equation (4) may be discontinued, and a straight line drawn down to the base of the dam. The vertical back hitherto assumed is only admissible for the full reservoir, as it fails to satisfy condition (1) when the reservoir is empty. This defect may be sufficiently remedied by giving the back a batter of 1 in 20,[1] commencing at the depth y_1. The increased thickness thus given to the dam upsets to some

[1] The straight batter is a close approximation, and nearer the correct form than the concave batter commonly adopted. To comply more nearly with condition (1) the back of the upper portion of the dam should be convex outwards.

extent the previous calculations for the outer face; but the error is small, and on the safe side, as it has the effect of bringing the curve of pressure for the full reservoir further into the middle third of the dam, and also compensates for the error involved in assuming the layers to be rectangular instead of trapezoidal.

To sum up the process of design: Equations (1) and (2) give the thickness of the top of the dam and its height above water-level; equations (3) or (3 a) determine the height of the rectangular portion of the profile, and equations (4) or (4 a) the curve of the outer face. A batter of 1 in 20 is then given to the inner face, and the profile is complete so far as condition (1) is concerned. It must then be tested by the graphic process described further on, and the stresses in the faces calculated, and below the depth at which they are found to exceed the safe limit, the thickness of the dam will have to be increased. One or two trials will generally suffice to determine the required thickness, which, however, may be approximately calculated by the following formulas:—

For the outer face—

$$x = \frac{1 \cdot 07}{\sqrt{2\theta}} \left[y + 0 \cdot 2857 \frac{\phi'}{S} y^2 + 0 \cdot 0808 \left(\frac{\phi'}{S} \right)^2 y^3 + 0 \cdot 0186 \left(\frac{\phi'}{S} \right)^3 y^4 \right] \quad (5)$$

x being the ordinate from the vertical axis to the outer face at any depth y; θ the specific gravity of the masonry; ϕ' the weight of a cubic unit of masonry; and S the stress allowed in the faces of the dam.

For the inner face—

$$z = 0 \cdot 06\, x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where z is the ordinate from the vertical axis to the inner face, and x the corresponding ordinate of the outer face calculated by equation (5).

Equation (5) is simply Mr. Pelletreau's formula multiplied by 1·07, in order to provide for the increased stress due to the obliquity of the resultant, and it may be noted that it is only applicable in the manner mentioned above, that is, for the portion of the outer face of the dam below the level at which the stress in that face is found to exceed the limit S.

The depth at which it becomes necessary to provide for condition (2) depends upon the weight of the masonry, and the permissible stress, and will usually be found to be about $0 \cdot 75 \frac{S}{\phi'}$ for the outer face, and $1 \cdot 10 \frac{S}{\phi'}$ for the inner face.

It may therefore be taken as a general rule that the dimensions of small dams, and of the upper part of large dams, are determined by condition (1) alone; and that unless the height exceeds about $0 \cdot 75 \frac{S}{\phi'}$ condition (2), *i.e.* the question of stress, does not enter into the calculations.

The use of equation (4) is the only part of the process which is at all lengthy; but it is very simple, and the Author thinks will be found to take less time, and give less trouble than a continued repetition of the graphic process, in which the areas and centres of gravity of the different layers into which the profile is divided, have to be re-calculated for each trial alteration. The accumulation of lines on the paper is also a source of much inconvenience.

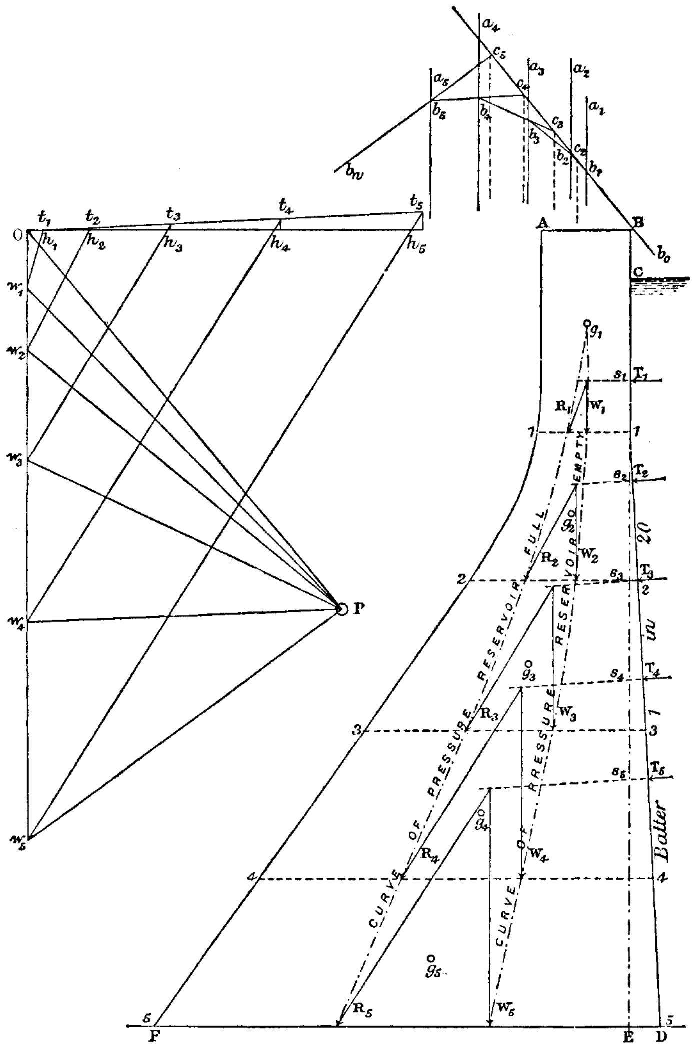
II.

A profile having been drawn in the manner described in Part I, its stability may be tested by the following partly graphic and partly analytical process—

Let the profile A B C D E F (Fig. 4) be divided into a convenient number of zones by the horizontal sections 1—1, 2—2, 3—3, &c.; and let $g_1\, g_2\, g_3$ . . . be the centres of gravity of the corresponding zones. Through $g_1\, g_2\, g_3$ . . . draw the vertical lines $a_1\, b_1$, $a_2\, b_2$, $a_3\, b_3$ . . . On the vertical line of the force-polygon set off to any convenient scale, $O w_1$, $w_1\, w_2$, $w_2\, w_3$ . . . . equal to the weights of the zones 1 2 3 . . . ., and from any pole P draw the radial lines PO $P w_1$ $P w_2$ $P w_3$ &c. In the vertical line a_1 take any point b_1, and through it draw $b_0\, b_1$ parallel to PO, and produce it indefinitely. Then draw $b_1\, b_2$ parallel to $P w_1$, $b_2\, b_3$ parallel to $P w_2$, and so on; the last line ($b_5\, b_n$ in the figure) being parallel to the last radial line of the force-polygon. Produce the sides of the polygon $b_2\, b_3$, $b_3\, b_4$ &c., to their intersections $c_2\, c_3\, c_4$ . . . with $b_0\, b_1$ produced, and through $c_2\, c_3\, c_4$ . . . draw vertical lines (dotted in the figure) down to the corresponding sections of the profile. This determines the positions of the resultant forces $W_1\, W_2\, W_3$ . . ., acting at the different sections when the reservoir is empty. Thus the force W_1 at section 1—1 is equal to the weight of the first zone, and acts through b_1. The force W_2 on section 2—2 is equal to the weights of the first and second zones, and acts through c_2, and so on for the other sections. The line joining the points where $W_1\, W_2\, W_3$, &c., cut the sections 1—1, 2—2, 3—3, &c., is the "curve of pressure" for the empty reservoir.

To find the curve of pressure for the full reservoir, it is only requisite to determine the thrust of the water T, due to the depth

FIG. 4.



Scale of profile $= \frac{1}{250}$.
Scale of forces 150,000 kilos. = 1 inch. $\theta = 2 \cdot 3$.

of each section below the water-level, and draw the resultant R of W and T. This may be done in the following manner:—

From the point O in the force-polygon draw a horizontal line, and on it set off (to the same scale as used for the weights) Oh_1 Oh_2 Oh_3 . . ., equal to the horizontal thrusts of the water due to the depths of the sections 1—1, 2—2, 3—3 . . . below the water-level, and through h_1 h_2 h_3 . . . draw the verticals $h_1 t_1$ $h_2 t_2$ $h_3 t_3$ . . . From O draw Ot_1 in a direction perpendicular to the back of the first zone, cutting $h_1 t_1$ at t_1 (in this case Ot_1 is horizontal, and t_1 coincides with h_1). Next draw $t_1 t_2$ perpendicular to the back of the second zone, and cutting $h_2 t_2$ at t_2, and so on. Then Ot_1 Ot_2, &c., give the magnitude and direction of the thrusts T_1 T_2, &c., for the corresponding sections. To find the position of these thrusts set off Cs_1, equal to two-thirds of C1; Cs_2 equal to two-thirds of C2, &c. Project these points horizontally to the back of the dam, and through the points of intersection so obtained draw T_1 parallel to Ot_1; T_2 parallel to Ot_2, &c. Produce T_1 T_2 T_3 to meet W_1 W_2 W_3, and through their intersection draw R_1 parallel to $t_1 w_1$, R_2 parallel to $t_2 w_2$, &c.

The line joining the points of intersection of R_1 R_2 R_3 . . . and the sections 1—1 2—2 3—3 . . . is the curve of pressure for the full reservoir.

It will now be seen whether the curves of pressure fall, as they should do, within the midde third of the thickness of the dam. This is an essential condition for the full reservoir, as any tension in the back of the dam might cause cracks in the mortar through which water would be admitted to the interior of the work. In such a case there would be an internal bursting-pressure which would tend to diminish the stability of the dam. For the empty reservoir, condition (2) is of less importance, and for this reason the method of design given in Part I has been devised in such a manner, that the curve of pressure for the full reservoir shall always be well within the middle third.[1]

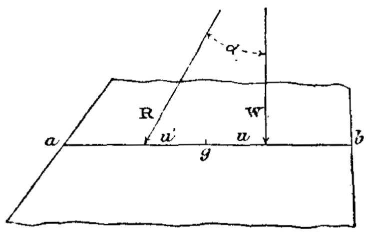
It now remains to calculate the stresses in the faces of the dam, for which purpose a general formula may be obtained.

Taking first the case of the empty reservoir, let $a\,b$ (Fig. 5) be any horizontal section of length $a\,b = l$, and width unity. Let W be the resultant pressure on $a\,b$, acting at a distance u from the

[1] In Fig. 4 the curve of pressure for the full reservoir is outside the middle third between sections 1 and 2, but this is only an error in the drawing, the distance between the sections having been taken greater than would be the case in practice, in order to make the diagram more distinct.

centre of gravity g of the section. The force W may be considered as causing in the section $a\,b$ a uniform compressive stress $= \frac{W}{l}$, and in addition to this a bending moment $= W\,u$, which produces compression at the edge nearest W, and a tension of the same amount at the opposite edge. The combined effect of these two stresses is the total stress S in the face of the dam. The stress at a or b, due to the bending moment $W\,u$, is given by the well-known formula $\frac{W u}{I} \times \frac{l}{2}$, where I is the moment of inertia of the section. The section being rectangular, $I = \frac{l^3}{12}$. Substituting this value,

FIG. 5.



and taking the algebraical sum of the two stresses, the expression for the stress at a or b becomes—

$$S = \frac{W}{l} \pm \frac{12\,W\,u\,l}{2\,l^3}$$

$$= \frac{W}{l}\left(1 \pm \frac{6\,u}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

With the plus sign this formula gives the stress at the edge nearest W, and with the minus sign the stress at the opposite edge. A minus value of S indicates tension, but this of course only occurs when the resultant force is outside the middle third.

For the full reservoir equation (7) requires a modification. It was formerly the custom, in calculating the stresses due to an oblique force, to consider only the vertical component of that force; but it has been shown by Mr. Bouvier that the stresses calculated in that manner are considerably below their correct value, and that instead of the vertical component, it is the force itself, divided by the cosine of the angle which it makes with the vertical, that must be taken as the effective force. Thus in Fig. 5,

U 2

if R is the resultant force acting at the section $a\,b$ when the reservoir is full, u' its distance from the centre of gravity of the section, and a the angle which it makes with the vertical, equation (7) becomes

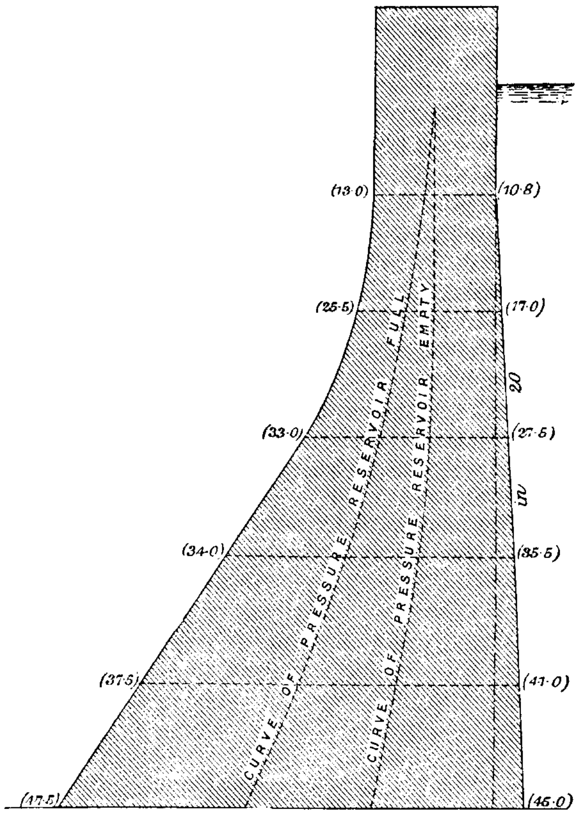
$$S = \frac{R}{l \cos a}\left(1 \pm \frac{6\,u'}{l}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

Owing to the difficult nature of the experiment, the compressive strength of mortar cannot be determined with exactitude. For all practical purposes, however, the compressive strength of mortar made of good hydraulic lime or cement, in the proportions that should be used in a dam, may be taken at ten times its tensile strength; so that a factor of safety of $\frac{1}{10}$ gives the simple rule *that the safe compressive stress is equal to the ultimate tensile strength of the mortar.* In order that the joints of the masonry may be watertight, the quality of the mortar is practically determined by the condition that the interstices of the sand should be completely filled up by the lime or cement, the required proportion of lime being ascertained by the amount of water that can be poured into a given quantity of dry sand. In some experiments which the Author made in this manner, the proportion of sand to water averaged about $2\frac{1}{3}$ to 1. The tensile strength of mortar made in this proportion with good hydraulic lime would probably not be less than 140 lbs. per square inch at the end of twelve months; but it must be remembered that the strength of mortar depends as much on the nature of the sand as on the quality of the lime or cement itself. Mr. J. Grant, M. Inst. C.E., quotes a case where two sands not much unlike each other gave results that differed by 50 per cent.

In the Paper already mentioned, Mr. Bouvier considers that in dams built of rubble masonry, with good hydraulic mortar, the permissible compressive stress may be taken at 10 kilograms per square centimetre (140 lbs. per square inch) at the completion of the work, and that later on, as the mortar attains its ultimate strength, this pressure may be raised to 14 kilograms per square centimetre (200 lbs. per square inch). This may be effected by limiting the depth of water, say for the first twelve or eighteen months, and then allowing it to rise by degrees to its final level.

It has been mentioned that in designing a dam in the manner described in Part I, the curve of pressure for the empty reservoir will sometimes be found to fall outside the middle third of the thickness. In this case the stress calculated by equation 7 will have a minus sign, indicating the tension in the outer face. This tension must be resisted either by the tensile strength of the mortar, or by the adhesion of the mortar to the stone. Assuming these two resistances to be equal, there would be no danger in allowing a tensile stress in the outer face of the dam equal to one-tenth of the ultimate tensile strength of the mortar.

FIG. 6.



NAYA DAM.

The figures in brackets give the stresses in lbs. per square inch in the outer face for the full reservoir, and in the inner face for the empty reservoir. $\theta = 2 \cdot 6$.

Scale $\frac{1}{125}$.

The specific gravity of rubble masonry is approximately equal to two-thirds of the specific gravity of the stone plus one-third of the specific gravity of the mortar. The quantity of stone required for a rubble-masonry dam may be estimated in the same way, there being about two-thirds of a cubic yard of solid stone to every cubic yard of masonry.

Fig. 6 is a section of the Naya dam lately built by the Author for the Rio Tinto Mining Company. It affords an illustration of the fact that the stress which the masonry is capable of supporting does nor necessarily influence the design, much less can it be made the sole basis of the calculation. The greatest stress in this dam is $47\frac{1}{2}$ lbs. to the square inch, whilst the mortar ($2\frac{1}{2}$ of sand to 1 of Portland cement) is capable of bearing with safety more than three times that amount. The curves of pressure towards the base coincide very nearly with lines limiting the middle third of the section, so that it would be impossible to reduce the area of the profile, and so increase the stresses, without at the same time bringing the curves of pressure outside their proper limits. On the other hand, to have reduced the quantity of cement so as to make its resistance suit the stresses, would have made the mortar porous, which in this case it was important to avoid, as the dam is destined to retain water containing large quantities of sulphuric acid. The joints of the inner face of this dam are pointed with neat cement.

In conclusion the Author would suggest the use of the metre and kilogram as units in designing a dam. The calculations are much easier than when English measures are used.

The Paper is accompanied by several diagrams, from which the Figs. in the text have been prepared.